W47. In every triangle ABC is true the inequality:

i). $4\sum \tan^2\frac{A}{2}\tan^2\frac{B}{2} - 3\sum \tan^3\frac{A}{2}\tan^3\frac{B}{2} \le 1$

ii). $2\sum \tan^2\frac{A}{2} + 9\sum \tan^2\frac{A}{2}\tan^2\frac{B}{2} \le 1$

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As it can be seen from further both inequalities should be corrected as follows:

(**i**) $4\sum \tan^2\frac{A}{2}\tan^2\frac{B}{2} - 3\sum \tan^3\frac{A}{2}\tan^3\frac{B}{2} \ge 1;$

(**ii**) $2\sum \tan^2\frac{A}{2} + 9\sum \tan^2\frac{A}{2}\tan^2\frac{B}{2} \ge 5.$

(counterexample for original (**ii**): **for** $A = B = C = \frac{\pi}{3}$ we have

$2\sum \tan^2\frac{A}{2} + 9\sum \tan^2\frac{A}{2}\tan^2\frac{B}{2} = 2\cdot 3\cdot\left(\frac{1}{\sqrt{3}}\right)^2 + 9\cdot 3\cdot\left(\frac{1}{\sqrt{3}}\right)^4 = 5$).

Since $\tan\frac{A}{2} = \frac{r}{s-a}$, where r and s be inradius and semiperimeter of $\triangle ABC$

and $r^2 = \frac{(s-a)(s-b)(s-c)}{s}$ then $I := 4\sum \tan^2\frac{A}{2}\tan^2\frac{B}{2} - 3\sum \tan^3\frac{A}{2}\tan^3\frac{B}{2} =$

$4\sum \frac{r^4}{(s-a)^2(s-b)^2} - 3\sum \frac{r^6}{(s-a)^3(s-b)^3} = \frac{4s\sum(s-c)^2 - 3\sum(s-c)^3}{s^3}$

and $J := 2\sum \frac{r^2}{(s-a)^2} + 9\sum \frac{r^4}{(s-a)^2(s-b)^2} = \frac{2}{s}\sum \frac{(s-b)(s-c)}{s-a} + \frac{9}{s^2}\sum(s-c)^2.$

Let $x := s-a, y := s-b, z := s-c, p := xy+yz+zx, q := xyz$. Then, assuming $s = 1$ (due homogeneity of I and J), we obtain $x,y,z > 0, x+y+z = 1$, $\sum(s-c)^2 = 1-2p$, $\sum(s-c)^3 = 1+3q-3p, \sum \frac{(s-b)(s-c)}{s-a} = \frac{1}{(s-a)(s-b)(s-c)}\sum(s-b)^2(s-c)^2 =$

$\frac{1}{q}\sum y^2z^2 = \frac{p^2-2q}{q}$ and, therefore, $I = 4(1-2p) - 3(1+3q-3p) = p - 9q + 1 \ge 1$

because $p = (x+y+x)(xy+yz+zx) \ge 3\sqrt[3]{xyz}\cdot 3\sqrt[3]{x^2y^2z^2} = 9xyz = 9q$

by AM-GM inequality.

Also, noting that $p^2 = (xy+yz+zx)^2 \ge 3xyz(x+y+z) = 3q$ and

$3p = 3(xy+yz+zx) \le (x+y+z)^2 = 1$ we obtain

$J = 2\sum \frac{yz}{x} + 9\sum z^2 = \frac{2}{q}\sum y^2z^2 + 9\sum z^2 = \frac{2(p^2-2q)}{q} + 9(1-2p) =$

$\frac{2p^2}{q} + 5 - 18p \ge \frac{2p^2}{(p^2/3)} + 5 - 18p = 11 - 18p \ge 11 - 18\cdot\frac{1}{3} = 5.$